Mathematical Finance
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Assignment 11

Option pricing in Heston model

Let $(\Omega,\mathcal{F},\mathbb{F}=(\mathcal{F}_t)_{t\geq 0},\mathbb{Q})$ be a filtered probability space, carrying two independent Brownian motions B and W, adapted to $\mathbb{F}$. We will assume directly and for simplicity that $\mathbb{Q}$ is a risk-neutral measure. In Heston stochastic volatility model, the financial market consists of two assets. First a non-risky one whose price S^0 satisfies $S^0_t := \mathrm{e}^{rt},\ t\geq 0$, where the short rate $r\geq 0$ is a fixed constant, and a risky asset, whose price S has dynamics given by

$$\begin{cases} S_t = S_0 + \displaystyle\int_0^t rS_s\mathrm{d}s + \int_0^t S_s\sqrt{v_s}\mathrm{d}B_s,\ t\geq 0,\\ v_t = v_0 + \displaystyle\int_0^t b(\theta - v_s)\mathrm{d}s + \int_0^t \sigma\sqrt{v_s}\big(\rho\mathrm{d}B_s + \sqrt{1-\rho^2}\mathrm{d}W_s\big), \end{cases}$$

where the parameters b, θ and σ are positive, and where $\rho\in[-1,1]$ is the correlation coefficient between S and v. We emphasise that the above equation has a unique solution which defines then meaningfully both S and v.

1) We define, for any $t\geq 0$, $X_t := \log(S_t)$. Prove that

$$X_t = X_0 + \int_0^t \left(r - \frac{1}{2}v_s\right)\mathrm{d}s + \int_0^t \sqrt{v_s}\mathrm{d}B_s,\ t\geq 0.$$

2) Fix some time horizon $T\geq 0$. For any $0\leq t\leq T$, we define now the conditional characteristic function of X_T given $\mathcal{F}_t$, for any $u\in\mathbb{R}$, by

$$\psi_t(u,T) := \mathbb{E}^{\mathbb{Q}}\big[\mathrm{e}^{iuX_T}\big|\mathcal{F}_t\big],$$

where we recall that $i\in\mathbb{C}$ is the complex number such that $i^2=-1$. Explain why $\psi_t(u,T)$ is actually a function of t, X_t and v_t, that is to say that we can write

$$\psi_t(u,T) = \Psi(t,X_t,v_t,u,T),$$

for some function $\Psi : [0,T]\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}_+\longrightarrow\mathbb{C}$.

3) We assume that the function $\Psi(t,x,v,u,T)$ is smooth in its first 3 variables. Define now the following 2-dimensional processes

$$V_t := \begin{pmatrix} X_t\\ v_t\end{pmatrix},\ \text{and } \mathcal{B}_t := \begin{pmatrix} B_t\\ W_t\end{pmatrix}.$$

Notice that $\mathcal{B}$ is then a 2-dimensional Brownian motion. Verify that we can write

$$V_t = V_0 + \int_0^t \begin{pmatrix} r - \frac{1}{2}V_s^2\\ b(\theta - V_s^2)\end{pmatrix}\mathrm{d}s + \int_0^t \sqrt{V_s^2}\begin{pmatrix} 1 & 0\\ \sigma\rho & \sigma\sqrt{1-\rho^2}\end{pmatrix}\mathrm{d}\mathcal{B}_s,\ t\geq 0,$$

where for $i\in\{1,2\}$, V^i denotes the ith component of the vector V.

4) Using Itô's formula, prove that for any $u\in\mathbb{R}$

$$\begin{aligned}\mathrm{d}\Psi(t,X_t,v_t,u,T) = {}& \left(\frac{\partial\Psi}{\partial t}(t,X_t,v_t,u,T) + \left(r-\frac{1}{2}v_t\right)\frac{\partial\Psi}{\partial x}(t,X_t,v_t,u,T) + b(\theta - v_t)\frac{\partial\Psi}{\partial v}(t,X_t,v_t,u,T)\right)\mathrm{d}t\\ &+\left(\frac{1}{2}v_t\frac{\partial^2\Psi}{\partial x^2}(t,X_t,v_t,u,T) + \frac{\sigma^2}{2}v_t\frac{\partial^2\Psi}{\partial v^2}(t,X_t,v_t,u,T) + \sigma\rho v_t\frac{\partial^2\Psi}{\partial x\partial v}(t,X_t,v_t,u)\right)\mathrm{d}t\\ &+\sqrt{v_t}\left(\frac{\partial\Psi}{\partial x}(t,X_t,v_t,u,T) + \sigma\rho\frac{\partial\Psi}{\partial v}(t,X_t,v_t,u,T)\right)\mathrm{d}B_t\\ &+\sigma\sqrt{1-\rho^2}\sqrt{v_t}\frac{\partial\Psi}{\partial v}(t,X_t,v_t,u,T)\mathrm{d}W_t.\end{aligned}$$

5) Prove that, for any $u \in \mathbb{R}$, $(\Psi(t, X_t, v_t, u, T))_{0 \leq t \leq T}$ is an $(\mathbb{F}, \mathbb{Q})$-martingale.

6) Deduce that for any $u \in \mathbb{R}$, the function Ψ must satisfy the partial differential equation

$$\begin{cases} \dfrac{\partial \Psi}{\partial t}(t,x,v,u,T) + \Big(r - \dfrac{1}{2}v\Big)\dfrac{\partial \Psi}{\partial x}(t,x,v,u,T) + b(\theta - v)\dfrac{\partial \Psi}{\partial v}(t,x,v,u,T) + \dfrac{1}{2}v\dfrac{\partial^2 \Psi}{\partial x^2}(t,x,v,u,T) \\ \quad + \dfrac{\sigma^2}{2}v\dfrac{\partial^2 \Psi}{\partial v^2}(t,x,v,u,T) + \sigma\rho v\dfrac{\partial^2 \Psi}{\partial x \partial v}(t,x,v,u,T) = 0, \ (t,x,v) \in [0,T) \times \mathbb{R} \times \mathbb{R}_+, \\ \Psi(T,x,v,u,T) = \mathrm{e}^{iux}. \end{cases}$$

7) We are now looking for a solution of the previous PDE of the form

$$\Psi(t,x,v,u,T) = \exp\big(iux + A(u,t,T) + B(u,t,T)v\big),$$

where the maps $A, B : \mathbb{R} \times [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ are our new unknowns. Prove that such a function solves the PDE if and only A and B satisfy the system

$$\begin{cases} \dfrac{\partial A}{\partial t}(u,t,T) + riu + b\theta B(u,t,T) = 0, \text{ for } t \in [0,T), \ A(u,T,T) = 0, \quad (1) \\ \dfrac{\partial B}{\partial t}(u,t,T) + \dfrac{\sigma^2}{2}B^2(u,t,T) + (\sigma\rho iu - b)B(u,t,T) - \dfrac{u}{2}(u+i) = 0, \text{ for } t \in [0,T), \ B(u,T,T) = 0. \quad (2) \end{cases}$$

8) Fix a complex number $\alpha(u,T)$ such that[1]

$$\frac{\sigma^2}{2}\alpha^2(u,T) + (\sigma\rho iu - b)\alpha(u,T) - \frac{u}{2}(u+i) = 0.$$

Check that any function of the form

$$\frac{1}{C(u,t,T)} + \alpha(u,T),$$

solves Equation (2) if and only if the function C solves the equation

$$\frac{\partial C}{\partial t}(u,t,T) = \big(\sigma\rho iu - b + \sigma^2\alpha(u,T)\big)C(u,t,T) + \frac{\sigma^2}{2}, \text{ for } t \in [0,T), \ C(u,T,T) = -\frac{1}{\alpha(u,T)}. \quad (3)$$

9) Check that the solution to Equation (3) is given by

$$C(u,t,T) = \left(\frac{\sigma^2}{2\kappa(u,T)} - \frac{1}{\alpha(u,T)}\right)\mathrm{e}^{-\kappa(u,T)(T-t)} - \frac{\sigma^2}{2\kappa(u,T)}, \ t \in [0,T],$$

where

$$\kappa(u,T) := \sigma\rho iu - b + \sigma^2\alpha(u,T),$$

and deduce that

$$B(u,t,T) = \alpha(u,T) + \frac{2\kappa(u,T)\alpha(u,T)e^{\kappa(u,T)(T-t)}}{\sigma^2\alpha(u,T) - 2\kappa(u,T) - \sigma^2\alpha(u,T)\mathrm{e}^{\kappa(u,T)(T-t)}}.$$

10) Deduce finally that

$$A(u,t,T) = \big(riu + b\theta\alpha(u,T)\big)(T-t) + \frac{2b\theta}{\sigma^2}\log\left(\frac{\sigma^2\alpha(u,T) - 2\kappa(u,T) - \sigma^2\alpha(u,T)}{\sigma^2\alpha(u,T) - 2\kappa(u,T) - \sigma^2\alpha(u,T)\mathrm{e}^{\kappa(u,T)(T-t)}}\right),$$

where you should admit that the logarithm function log can be meaningfully extended to complex numbers.

[1] Notice that there are in general two such numbers $\alpha(u,T)$ given by

$$\alpha(u,T) = \frac{b - \sigma\rho iu \pm \sqrt{(\sigma\rho iu - b)^2 + u(u+i)\sigma^2}}{\sigma^2},$$

where the square-root of a complex number $x \in \mathbb{C}$ should be interpreted as any $y \in \mathbb{C}$ such that $y^2 = x$. However, we will not use the explicit value given above in the rest of the exercise.

Indifference pricing in a stochastic volatility model

We fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and consider a financial market in continuous-time, with one non-risky asset whose price S^0 is normalised to 1, and a risky asset with price process S defined by

$$\frac{\mathrm{d}S_t}{S_t} = \sigma(Y_t)\big(\lambda(Y_t)\mathrm{d}t + \mathrm{d}W_t^1\big), \tag{0.1}$$

where Y is an exogenous factor whose dynamics is given by

$$\mathrm{d}Y_t = \eta(Y_t)\mathrm{d}t + \gamma(Y_t)\big(\rho \mathrm{d}W_t^1 + \sqrt{1-\rho^2}\mathrm{d}W_t^2\big). \tag{0.2}$$

Here, $W := (W^1, W^2)^\top$ is a 2-dimensional $(\mathbb{F}, \mathbb{P})$–Brownian motion, ρ is a constant in $(-1, 1)$. The coefficients $\eta(y)$, $\gamma(y)$, $s\lambda(y)$, $s\sigma(y)$ are assumed to be such that the system of SDEs (0.1)–(0.2) has a unique strong solution. We will assume in addition that $\lambda(\cdot)$, $\sigma(\cdot)$ are bounded and that $\inf_{y\in\mathbb{R}}\{\sigma(y)^2 + \gamma^2(y)\} > 0$. We also assume that $\mathbb{F}$ is the $\mathbb{P}$-completed natural filtration of W.

We denote by $\mathcal{A}$ the admissible self-financing strategies, that is to say all the $\Delta \in \mathcal{L}^1(S, \mathbb{F}, \mathbb{P})$ such that $X^{0,\Delta}$ remains bounded from below (recall here that S^0 is contant equal to 1). We consider a European option written on the non-tradable asset Y

$$G := g(Y_T), \text{ where } g : \mathbb{R} \longrightarrow \mathbb{R} \text{ is continuous and bounded.}$$

We define the optimal investment problem

$$V^g(0, x, Y_0) := \sup_{\theta\in\mathcal{A}} \mathbb{E}^{\mathbb{P}}\Big[U\big(X_T^{x,\theta} - g(Y_T)\big)\Big], \text{where} U(x) := -\mathrm{e}^{-ax},\ a > 0,\ x \in \mathbb{R}.$$

The objective is to find an explicit expression for

$$p(0, x, y) := \inf\big\{p \in \mathbb{R} : V^g(0, x + p, y) \geq V^0(0, x, y)\big\}.$$

In the rest of the exercise, we extend all the above notations to the case where the time origin is some $t \geq 0$ instead of just 0, and introduce $V^g(t, x, y)$ and $p(t, x, y)$. Besides, we will assume that the map V^g is $C^{1,2}$ on $[0, T) \times \mathbb{R} \times \mathbb{R}$, and continuous on $[0, T] \times \mathbb{R} \times \mathbb{R}$, for all maps g as above.

1) Give an interpretation of p.

2) Show that there exists a positive map $F^g(t, y)$ such that

$$V^g(t, x, y) = -\mathrm{e}^{-ax}F^g(t, y), \text{for all } t \in [0, T],\ (x, y) \in \mathbb{R}^2,$$

and deduce that

$$p(t, x, y) = p(t, y) = \frac{1}{a}\log\left(\frac{F^g(t, y)}{F^0(t, y)}\right),\ (t, x, y) \in [0, T] \times \mathbb{R}^2.$$

3) We admit that the map $V^g(t, x, y)$ is a smooth solution to the PDE

$$\inf_{\theta\in\mathbb{R}}\big\{-\mathcal{L}^\theta V^g(t, x, y)\big\} = 0,\ (t, x, y) \in [0, T) \times \mathbb{R}^2,\ V^g(T, x, y) = U(x - g(y)),\ (x, y) \in \mathbb{R}^2.$$

where

$$\mathcal{L}^\theta V^g := \partial_t V^g + \eta\partial_y V^g + \frac{1}{2}\gamma^2\partial_{yy}V^g + \lambda\sigma\theta\partial_x V^g + \frac{1}{2}\sigma^2\theta^2\partial_{xx}V^g + \theta\rho\sigma\gamma\partial_{xy}V^g.$$

Show that $F^g(t, y)$ satisfies

$$\partial_t F^g + \eta\partial_y F^g + \frac{1}{2}\gamma^2\partial_{yy}F^g - \frac{(\lambda F^g + \rho\gamma\partial_y F^g)^2}{2F^g} = 0,\ (t, y) \in [0, T) \times \mathbb{R},\ F^g(T, y) = \mathrm{e}^{ag(y)},\ y \in \mathbb{R}.$$

4) Show that there exists $\delta \in \mathbb{R}$ such that the map $f^g(t, y) := (F^g(t, y))^{1/\delta}$ solves a *linear* PDE.

5) Deduce that

$$f^g(t,y) = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{(1-\rho^2)(ag(Y_T^{t,y}) - \frac{1}{2}\int_t^T \lambda^2(Y_u^{t,y})\mathrm{d}u)}\right],$$

where $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ with density

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} := \exp\left(-\rho \int_t^T \lambda(Y_s^{t,y})\big(\rho \mathrm{d}W_s^1 + \sqrt{1-\rho^2}\mathrm{d}W_s^2\big) - \frac{1}{2}\rho^2 \int_t^T \lambda(Y_s^{t,y})^2 \mathrm{d}s\right).$$

6) Introduce another measure $\hat{\mathbb{Q}}$ equivalent to $\mathbb{P}$ so as to write

$$p(t,y) = \frac{1}{a(1-\rho^2)} \log\left(\mathbb{E}^{\hat{\mathbb{Q}}}\big[\mathrm{e}^{a(1-\rho^2)g(Y_T^{t,y})}\big]\right).$$